Mixed evolutionary strategies imply coexisting opinions on networks

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An evolutionary battle-of-the-sexes game is proposed to model the opinion formation on networks. The individuals of a network are partitioned into different classes according to their unaltered opinion preferences, and their factual opinions are considered as the evolutionary strategies, which are updated with the birth-death or death-birth rules to imitate the process of opinion formation. The individuals finally reach a consensus in the dominate opinion or fall into (quasi)stationary fractions of coexisting mixed opinions, presenting a phase transition at the critical modularity of the multiclass individuals' partitions on networks. The stability analysis on the coexistence of mixed strategies among multiclass individuals is given, and the analytical predictions agree well with the numerical simulations, indicating that the individuals of a community (or modular) structured network are prone to form coexisting opinions, and the coexistence of mixed evolutionary strategies implies the modularity of networks.

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I. INTRODUCTION

Modeling the dynamics of opinion formation on a network is an important means to investigate the relationship between dynamical processes and topologies of complex networks. Opinion formation is comprehensibly achieved through the mutually convincing process among individuals of different opinions. In the traditional models of opinion formation, including the voter model, the Sznajd model, the Galam's majority rule, and so on, the individuals have no distinction between different opinions and/or the interactions of the opinion update in convincing processes are relatively simple and deterministic [1].

Game theory provides a versatile framework to model the interactions between adaptive individuals, where the individuals adapt their strategies/opinions to convince (or to be convinced by) other individuals' strategies and opinions to maximize their fitness or payoff [2]. In this paper, an evolutionary battle-of-the-sexes game (BSG) [3] is proposed to model the opinion formation on a network, where the diversity of individuals' opinion preferences are considered as corresponding payoff matrices of BSG and their factual opinions evolve with the evolutionary game strategies. During the convincing process of opinion formation, we consider two updating rules of the factual opinions: the birth-death (BD) process and the death-birth (DB) process, where the payoff- (fitness-) dependent selection is incorporated to mimic the persuading interactions among individuals.

Our main concern in this paper is the evolutionary-game-based opinion formation on the networks with community (or modular) structures [4,10]. We analytically and numerically show that not only the individuals' different preferences as well as their preference degrees (which are represented by the parameters of payoff matrices in the game) and also the modular network structure of individuals (whose modularity is partitioned according to their preferences) play a key role in the process of opinion and strategy formation during the game evolution. The individuals' factual opinions finally reach a consensus in the unique dominant opinion (the opinion dominance phase), or fall into (quasi)stationary fractions of coexisting mixed opinions (the opinion coexistence phase) whose phase transition criticality is determined by the coexistence stability of evolutionary strategies. In particular, the opinion coexistence phase reveals the community structural (or the modular) characteristics of network topology, where almost individuals' factual opinions are consistent with their preferences and partitioned into clusters of mixed coexisting opinions as different network communities.

The rest of this paper is organized as follows. In Sec. II we first make a brief introduction to the standard BSG as well as the generalized BSG on networks and formulate our model with two evolutionary BD and DB rules. Next, we give a general theoretical analysis in Sec. III and make the numerical verifications on the constructed networks with tunable community strengths in Sec. IV, which illustrate the dependence of phase transition criticality on the modularity network structure. Finally, we conclude the whole work in Sec. V.

II. MODEL DESCRIPTION

The standard BSG is an illustrational game to describe the convincing process [3]. One couple plan to go to the same event together, but each prefers a different event; i.e., they hold different opinions at the beginning: the wife prefers the opera while the husband is more willing to watch a boxing match. Mathematically, a BSG payoff matrix is written as

	opera	boxing
opera	$(2x,x)$	(0,0)
boxing	(0,0)	$(x,2x)$

where x is a positive number, which means if the wife convinces the husband of seeing the opera, then the wife will

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receive a payoff $2x$ and the husband will also receive x; similarly, the husband will receive the higher payoff $2x$ than the wife if the husband convinces his wife contrarily. However, if they cannot convince each other and go to different events, both of them will get nothing. The BSG provides a game theoretic solution to the opinion formation: two individuals hold different preferences but finally reach a consensus in their opinions and strategies, since both (opera, opera) and (boxing, boxing) are strict Nash equilibria of the BSG.

We consider the generalized l-strategy two-person BSG with the game strategies representing the finite different l opinions. Each individual prefers one of l candidate strategies and opinions and keeps unaltered. Formulate l different opinion preferences with payoff matrices $A_i=(a_{st}^i)_{l\times l}$, $1\leqslant i\leqslant l$, respectively, where a_{st}^i represents the payoff of an individual with preference of ith opinion to select opinion s when meeting another individual that selects opinion t. Thus A_i is a non-negative diagonal matrix, whose strictly largest diagonal entry is located at the ith row and ith column. If the consensus of the factual opinions is reached in accordance with one individual's preference, then he will get the maximum payoff or receive a payoff due to his compromise to follow his opponent's opinion. If the two individuals select different opinions as their respective strategies, both of them receive nothing.

We now put a step forward to play the game among a population of individuals, where the population structure is generally described by the network of contacts in the graph theory. Consider the evolutionary BSG on a directed network G with n vertices (individuals) and m links, where the individuals are partitioned into k $(1\leqslant k\leqslant l)$ classes due to their opinion preferences. We introduce the class-adjacency-strength matrix of G to describe the network structure of individual-class-leveled connections. We first construct a renormalized network C as follows: replace each class of individuals in the original network by a single vertex and map each link in the original network to a new link between the corresponding vertices of C. Thus C is a k-vertex network containing self-loops and multilinks. Define the class-adjacency-strength matrix $W=(w_{ij})_{k\times k}$ as the normalized adjacency matrix of C, where w_{ij} is the probability that an individual of class i meets an individual of class j along a randomly selected link.

In every round, each individual plays the BSG with his every out-reached neighbor and receives a payoff by averaging over all rewards from the game with his neighbors. The average payoff of individual with strategy j of class i is calculated as

$$u_i^j=\sum_{s=1}^{l}a_{js}^i\sum_{t=1}^{k}w_{it}x_t^s, \tag{1}$$

where x_i^j is the fraction of individuals with strategy j in class i and the sum $\sum_{t=1}^{k}w_{it}x_t^s$ is the expected frequency of individuals with strategy s who play the game with individuals of class i under the mean-field approximation. Therefore the average payoff u_i^j is regarded as the corresponding payoff that one individual receives from one opponent of the mixed strategies with frequencies $\sum_{t=1}^{k}w_{it}x_t^s$.

After one round of game playing, the convincing process takes place between a pair of individuals along a directed link (v_i,v_j) pointed from v_i to v_j: v_i convinces v_j, and v_j is convinced by v_i in next round. Analogous to the BD-DB process in the population dynamics [5], it is inclined to select a convincing individual v_i with a high average payoff. We consider the following two payoff-dependent selection rules.

(i) BD rule: the individual v_i is selected with the probability proportional to his averaged payoff. Then a randomly selected out-reached neighbor of v_i is considered to be substituted by the opinion of individual v_i.

(ii) DB rule: a randomly selected individual v_i updates his strategy as the coordinating opinion of one of his in-directed neighbors that is selected with the probability proportional to his averaged payoff.

Notice that the interacting graph and updating graph are unnecessarily the same [6]; i.e., an individual can play the game with a part of individuals as his interacting-neighbor set and replace his opinion and strategy according to another part of individuals as his updating-neighbor set, where the two sets are allowed to be different on some occasion. The difference between the interacting and updating graphs could result in complex asymptotic behaviors. Therefore we assume the interacting graph and the updating graph are not necessarily the same in the rest part of this paper.

III. ANALYTICAL RESULTS

Denote the interacting and updating graphs by G and G' (corresponding to the class-adjacency-strength matrices W and W', respectively), where n individuals are partitioned into k classes. In each class i, the population size is denoted by n_i $(\sum_{i=1}^{k}n_i=n)$ and n_i^j is the number of individuals in class i selecting the strategy/opinion j $(\sum_{j=1}^{l}n_i^j=n_i)$, letting the fraction of individuals of class i be $p_i=n_i/n$.

We first consider the case of the generalized BSG with the BD updating rule. In the birth step, recall the fraction of individuals with strategy j in class i, $x_i^j=n_i^j/n_i$, and thus an individual with strategy h of class t is selected with probability

$$r_t^h=\frac{x_t^hu_t^h}{\sum_{v=1}^{k}p_v\sum_{s=1}^{l}x_v^su_v^s}=\frac{x_t^hu_t^h}{\bar{u}},$$

where $\bar{u}=\sum_{v=1}^{k}p_v\sum_{s=1}^{l}x_v^su_v^s$ is the average payoff of all individuals. In the death step, an individual with strategy j of class i is randomly selected to follow the strategy h with probability $x_i^j\sum_{t=1}^{k}w_{it}'r_t^h$, where $W'=(w_{ij}')_{k\times k}$ is the class-adjacency-strength matrix of the updating network and $\sum_{t=1}^{k}w_{it}'r_t^h$ indicates the probability that an individual of class i is a randomly selected neighor of an individual with strategy h. The increment of x_i^j is due to the process of replacing a non-j-strategy individual of class i by strategy j, and the decrement of x_i^j is due to replacing a j-strategy individual of class i by nonstrategy j. Therefore, we have the rate equation

$$\dot{x}_i^j = \sum_{h=1}^{l} (1-\delta_{j,h}) x_i^h \sum_{t=1}^{k} w'_{it} r_t^j - \sum_{h=1}^{l} (1-\delta_{j,h}) x_i^j \sum_{t=1}^{k} w'_{it} r_t^h$$
$$= \sum_{h=1}^{l} x_i^h \sum_{t=1}^{k} w'_{it} r_t^j - \sum_{h=1}^{l} x_i^j \sum_{t=1}^{k} w'_{it} r_t^h$$
$$= \sum_{t=1}^{k} w'_{it} r_t^j - x_i^j \sum_{t=1}^{k} w'_{it} \sum_{h=1}^{l} r_t^h = \sum_{t=1}^{k} w'_{it} \left(r_t^j - x_i^j \sum_{h=1}^{l} r_t^h \right), \quad (2)$$

where $\delta_{i,j}$ is 1 if i equals j and 0 otherwise. It is easy to verify the equilibria $(x_i^j)^*$ to satisfy

$$\sum_{t=1}^{k} w'_{it} \left(r_t^{j^*} - x_i^{j^*} \sum_{h=1}^{l} r_t^{h^*} \right) = 0. \quad (3)$$

One obvious equilibrium yielded from $r_t^j - x_i^j \sum_{h=1}^{l} r_t^h = 0$ is a homogeneous strategy distribution for all the classes of individuals; i.e., for each strategy j, the fraction of strategy j are identical among any class of individuals. Omitting the subscript index which stands for an individual's class, we have $x^j = r^j$ and $u^j = \bar{u}$. Therefore, the homogeneous strategy distribution (if the solution exists) shows that individuals with any strategy of any class receive the equal average payoff.

We next analyze the stability of the equilibrium $(x_i^j)^*$ of Eq. (2). We consider any perturbation added to the equilibrium due to the strategy updating and give a stability condition of stationary strategy distribution around the equilibrium.

Rewrite Eq. (2) as

$$\dot{x}_i^j = \sum_{t=1}^{k} w'_{it} r_t^j - x_i^j \sum_{t=1}^{k} w'_{it} \sum_{h=1}^{l} r_t^h \triangleq - c_i (x_i^j - q_i^j), \quad (4)$$

where $c_i = \sum_{t=1}^{k} w'_{it} \sum_{h=1}^{l} r_t^h$ and $q_i^j = \sum_{t=1}^{k} w'_{it} x_t^j u_t^j / \sum_{t=1}^{k} w'_{it} \sum_{h=1}^{l} x_t^h u_t^h$. Assume that the stationary strategy distribution is established and those individuals of class i with strategy h follow the BD rule to adopt strategy j, where the updating of the individual brings the same increment and decrement of x_i^j and x_i^h with the quantity $\frac{1}{n_i}$. Since each payoff matrix A_i is non-negative, $c_i \geqslant 0$ holds deterministically. To decay all such a pair of perturbations as $(dx_i^j, -dx_i^h)$, for any i, j, and h at the equilibrium $(x_i^j)^*$, it requires

$$\frac{\partial q_i^j}{\partial x_i^j} - \frac{\partial q_i^j}{\partial x_i^h} < 1. \quad (5)$$

Next, we discuss the DB updating rule in a similar way briefly. The decrement of x_i^j in the DB process is due to replacing the randomly selected individual of class i with strategy j, and the increment of x_i^j is due to the random selected individual with strategy j as a neighbor of the individual of class i. Recalling that p_i is the fraction of individuals of class i that equal the probability of being randomly selected in the birth step, we write the rate equation of DB process as

$$\dot{x}_i^j = - p_i \left(x_i^j - \frac{\sum_{t=1}^{k} w'_{it} x_t^j u_t^j}{\sum_{t=1}^{k} w'_{it} \sum_{h=1}^{l} x_t^h u_t^h} \right) = - p_i (x_i^j - q_i^j).$$

The stationary equilibria also satisfy $x_i^{j^*} = q_i^j$, where q_i^j is defined as the same as in Eq. (4). Note that p_i is positive constant for a given population; thus, the stability condition can be yielded similarly as Condition (5).

A. Case of common preference BSG

We first consider the simplest case of the population having the common preference in the generalized BSG, where the class number $k=1$. Denote the payoff matrix $A = \mathrm{diag}(a_1, \ldots, a_l)$, and assume the rescaled payoff matrix entries satisfy $\sum_{j=1}^{l} 1/a_j = 1$. Here, the superscript indices of the parameters standing for the individual classes are omitted for simplification. We have the coexisting opinion distribution $x^* = (1/a_1, \ldots, 1/a_l)$ and $q_i = a_i x_i^2 / \sum_{j=1}^{l} a_j x_j^2$. Therefore $\frac{\partial q_i}{\partial x_i} - \frac{\partial q_i}{\partial x_j}|_{x^*} = 2 > 1$, which indicates the instability of the coexisting opinions.

Consider the consensus opinion equilibrium $x^* = (0, \ldots, 0, 1, 0, \ldots, 0)$ with only one entry being 1 and the other entries being 0, which means that the corresponding opinion is dominant among the whole population and adopted by all individuals as their final factual opinion. One can directly verify that $\frac{\partial q_i}{\partial x_i} - \frac{\partial q_i}{\partial x_j}|_{x^*} = 0 < 1$ to suffice the stability condition of the stationary distribution. The consensus of factual opinions of a population of well-mixed homogeneous individuals shows the population structure trivially constitutes one community as a whole. Provided that the consensus of opinion formation is reached in the nonpreferred one (whose corresponding diagonal entry is not the largest), the stationary distribution for a consensus opinion is not the globally optimum in the sense of maximizing the game utility, but the consensus opinion distribution has a locally stable optimal structure. Actually, the (quasi)stationary opinion distribution is determined by the states of the initial opinions and the opinion with the maximum payoff under the initial states will convince all other opinions when the network game evolves. Therefore, if the initial random opinion distribution is uniform, the preference opinion will be finally formed. If more than one opinion has equal payoff, there exist bistable stationary opinion distributions. In realizations of numerical simulations, owing to the randomness of BD-DB process, the stationary distribution could be driven away from the equilibrium point due to stochastic noises in the payoff-dependent selection.

B. Case of bipreference BSG

We next consider the case of class number $k=2$; i.e., there exist two opinion preferences among the individuals on the network. Assume the interacting and updating graphs G and G' with the class-adjacency-strength matrices

$$W = \begin{pmatrix} 1-p_1 & p_1 \\ p_2 & 1-p_2 \end{pmatrix} \quad \text{and} \quad W' = \begin{pmatrix} 1-p_1' & p_1' \\ p_2' & 1-p_2' \end{pmatrix}$$

and the payoff matrices for the two classes,

$$A_1 = \begin{pmatrix} b_1 & 0 \\ 0 & 1-b_1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 1-b_2 & 0 \\ 0 & b_2 \end{pmatrix},$$

where b_1 and b_2 (larger than 0.5) stand for the preference degrees of classes 1 and 2, respectively. Due to the normalized conditions $x_i^1 + x_i^2 = 1$ for any class i, we denote by x_1 and x_2 the fraction of individuals with the first opinion of each class for simplicity. Substituting the above-mentioned parameters and Eq. (1) into Eq. (4), we have

$$q_1 = \frac{A_1x_1^2 + B_1x_1x_2 + C_1x_2^2}{D_1x_1^2 + E_1x_1x_2 + F_1x_2^2 + G_1x_1 + H_1x_2 + I_1},$$

$$q_2 = \frac{A_2x_1^2 + B_2x_1x_2 + C_2x_2^2}{D_2x_1^2 + E_2x_1x_2 + F_2x_2^2 + G_2x_1 + H_2x_2 + I_2},$$

where

$$A_1 = b_1(1-p_1)(1-p_1'), \quad A_2 = b_1(1-p_1)p_2',$$

$$B_1 = b_1p_1(1-p_1') + (1-b_2)p_2p_1',$$

$$B_2 = b_1p_1p_2' + (1-b_2)p_2(1-p_2'),$$

$$C_1 = (1-b_2)(1-p_2)p_1', \quad C_2 = (1-b_2)(1-p_2)(1-p_2'),$$

$$D_1 = (1-p_1)(1-p_1'), \quad D_2 = (1-p_1)p_2',$$

$$E_1 = p_1(1-p_1') + p_2p_1', \quad E_2 = p_1p_2' + p_2(1-p_2'),$$

$$F_1 = (1-p_2)p_1', \quad F_2 = (1-p_2)(1-p_2'),$$

$$G_1 = -(1-b_1)(2-p_1)(1-p_1') - b_2p_2p_1',$$

$$G_2 = -(1-b_1)(2-p_1)p_2' - b_2p_2(1-p_2'),$$

$$H_1 = -(1-b_1)p_1(1-p_1') - b_2(2-p_2)p_1',$$

$$H_2 = -(1-b_1)p_1p_2' - b_2(2-p_2)(1-p_2'),$$

$$I_1 = (1-b_1)(1-p_1') + b_2p_1', \quad I_2 = (1-b_1)p_2' + b_2(1-p_2'). \tag{6}$$

One can directly verify two consensus opinion distributions by $x_1^* = x_2^* = 1$ or $x_1^* = x_2^* = 0$, and another stationary equilibrium is implied by $x_i = q_i$ (the equation set is cubic), in terms of which the stability condition is reduced to $\frac{\partial q_i}{\partial x_i} < \frac{1}{2}$.

In more general cases, solving $x_i^j = q_i^j$ yields the possible stationary distributions as implicit functions of the payoff matrices and the class-adjacency-strength matrices of interacting-updating graphs. We therefore numerically solve the rate equation and use Monte Carlo simulations of the BD-DB processes to verify the theoretical analysis. Note that the BD-DB updating rules yield identical stationary distributions (but in different time scales); therefore, in the next section we only visualize the numerical simulation results with the DB rule.

IV. NUMERICAL SIMULATIONS

We consider a simple illustrational network model preserving the numbers of vertices and links with tunable class-adjacency-strength matrices [7], where the individuals are partitioned into two equal-sized classes. The individuals prefer two different opinions, but with an identical preference degree, and randomly select their factual opinions at the initial time. At every time step, each individual plays one round of the BSG with his neighbors of the interacting graph and receives his payoff. Then one individual is picked, and his strategy is updated according to the DB rule among the population on the updating graph [6]. As shown in Fig. 1, we calculate and record the time series of fraction x_1 and x_2 (corresponding to the individuals in classes 1 and 2, respectively, who prefer the first opinion) in three simulation paths. The results are obtained by the agent-based evolutionary game on the constructed original networks, which agree well with the solution to the rate equations as analytically predicted on the corresponding renormalized networks (the dotted line in Fig. 1).

To verify the coexistence stability condition of the mixed opinions, we substitute the parameters [of Fig. 1(a)] $p_1 = p_1' = \frac{1}{13}$, $p_2 = p_2' = \frac{1}{16}$, and $b_1 = b_2 = \frac{3}{4}$ into Eq. (6) and have

$$q_1 = \frac{0.6391x_1^2 + 0.0545x_1x_2 + 0.0180x_2^2}{0.8521x_1^2 + 0.0758x_1x_2 + 0.0721x_2^2 - 0.4474x_1 - 0.1295x_2 + 0.2885},$$

$$q_2 = \frac{0.0433x_1^2 + 0.0183x_1x_2 + 0.2197x_2^2}{0.0577x_1^2 + 0.0634x_1x_2 + 0.8789x_2^2 - 0.0740x_1 - 1.3635x_2 + 0.71875}.$$

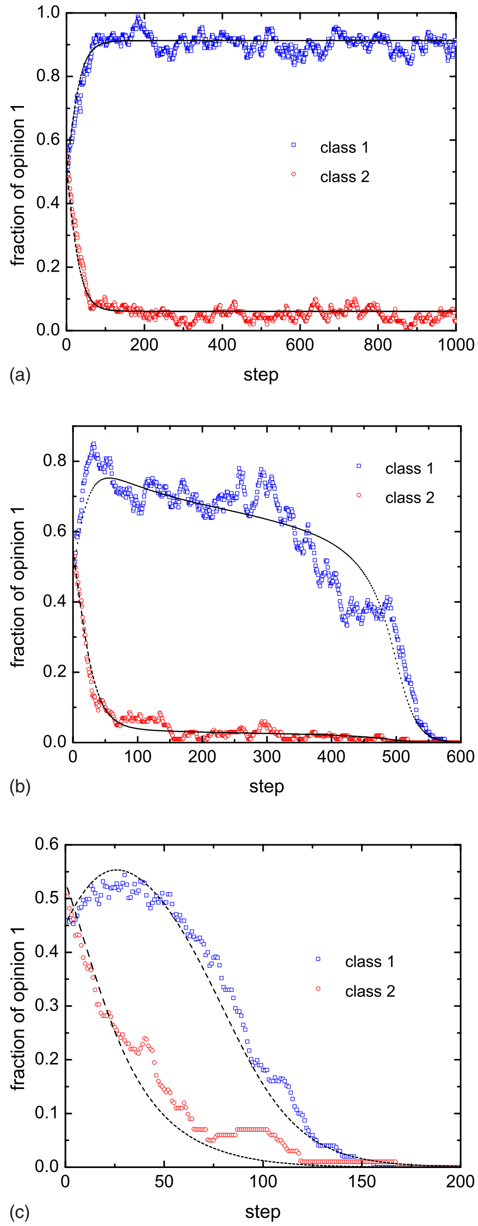

FIG. 1. (Color online) The opinion formation implemented by the evolutionary BSG with the DB updating rule on the networks with two equal-sized classes of individuals. The plotted curves are the time series of fractions of opinion 1 among the individuals in class 1 (blue boxes) and class 2 (red circles), respectively, which fit the analytical prediction (dotted lines) well. The class-adjacency-strength matrices and modularities of the interacting and updating graphs are (a) $W=\begin{pmatrix} 12/13 & 1/13 \\ 1/16 & 15/16 \end{pmatrix}$, $\mathcal{M}=0.430$; (b) $W=\begin{pmatrix} 6/7 & 1/7 \\ 1/16 & 15/16 \end{pmatrix}$, $\mathcal{M}=0.397$; (c) $W=\begin{pmatrix} 3/4 & 1/4 \\ 1/16 & 15/16 \end{pmatrix}$, $\mathcal{M}=0.344$. Here, the interacting and updating graphs are identical. Each class of individuals (whose population size $n_1=n_2=100$) has the preference degree $b_1=b_2=0.75$. At every step, the data point is averaged over ten rounds of realization of the BSG. The analytical prediction is yielded from the numerical solution to the corresponding rate equations with a proper integral step scale $1/nl$, whose growth or decay rate is equivalent to the average increase or decrease of x_i^j in one simulation step under the DB update.

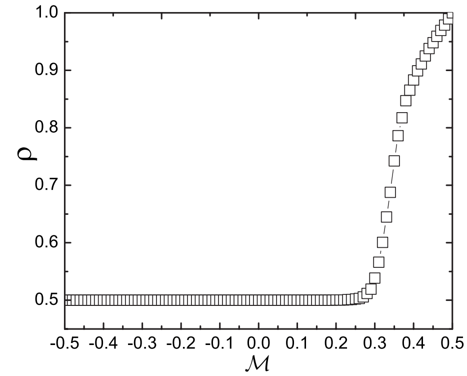

FIG. 2. The order parameter ρ (the percentage of satisfactory individuals whose factual opinions are consistent with their preferences) as a function of modularity $\mathcal{M}$. The class number $k=2$, the population sizes in each class $n_1=n_2=100$, and the preference degrees $b_1=b_2=0.75$. Each data point is obtained by averaging 1000 different realizations.

Therefore the coexisting opinion distribution is $x_1^*=0.9129$ and $x_2^*=0.0609$. One can directly verify that the stability conditions $\frac{\partial q_1}{\partial x_1}=0.2624<\frac{1}{2}$ and $\frac{\partial q_2}{\partial x_2}=0.1866<\frac{1}{2}$ hold. And for the other cases of Figs. 1(b) and 1(c), the stable stationary distribution is $x_1^*=x_2^*=0$, whose stability condition can be verified similarly.

One of our main concerns is finding the correlation between the population topological structures and opinion dynamical behaviors, where the coexistence of mixed opinions is directly affected by the modularity (community) of network structure among the multiclass individuals. To quantitatively describe the strength of community structure for a given partition of multiclass individuals into network modules, following the first definition of modularity introduced by Newman *et al.*, we adopt the modularity $\mathcal{M}$ of the vertex partition on a directed network defined as [9]

$$\mathcal{M}=\sum_{c=1}^{k}\left[\frac{l_c}{m}-\left(\frac{d_c}{m}\right)^2\right], \tag{7}$$

where k is the number of classes, m is the number of the links in the network, l_c is the number of links connecting the vertices in class c, and d_c is the sum of out-degrees of the vertices in class c. $\mathcal{M}=0$ indicates a random partition of vertices, and a large modularity $\mathcal{M}$ for a given individual partition shows the densely clustering community structure of congeneric individuals that hold the same preference. Using the tunable community-strength (noted by class-adjacency-strength matrices W) directed network model [7], we construct networks with

$$W=\begin{pmatrix} 1-p_1 & p_1 \\ p_2 & 1-p_2 \end{pmatrix},$$

whose modularity is directly correlated with the parameters p_1 and p_2 in a very simple linear form as $\mathcal{M}=\frac{1}{2}-\frac{p_1+p_2}{2}$ [9].

As shown in Figs. 1(a) and 1(b) where the modularity $\mathcal{M}=0.430$ and 0.397, respectively, the final opinion distributions with different parameters p_1 and p_2 fall into the opinion

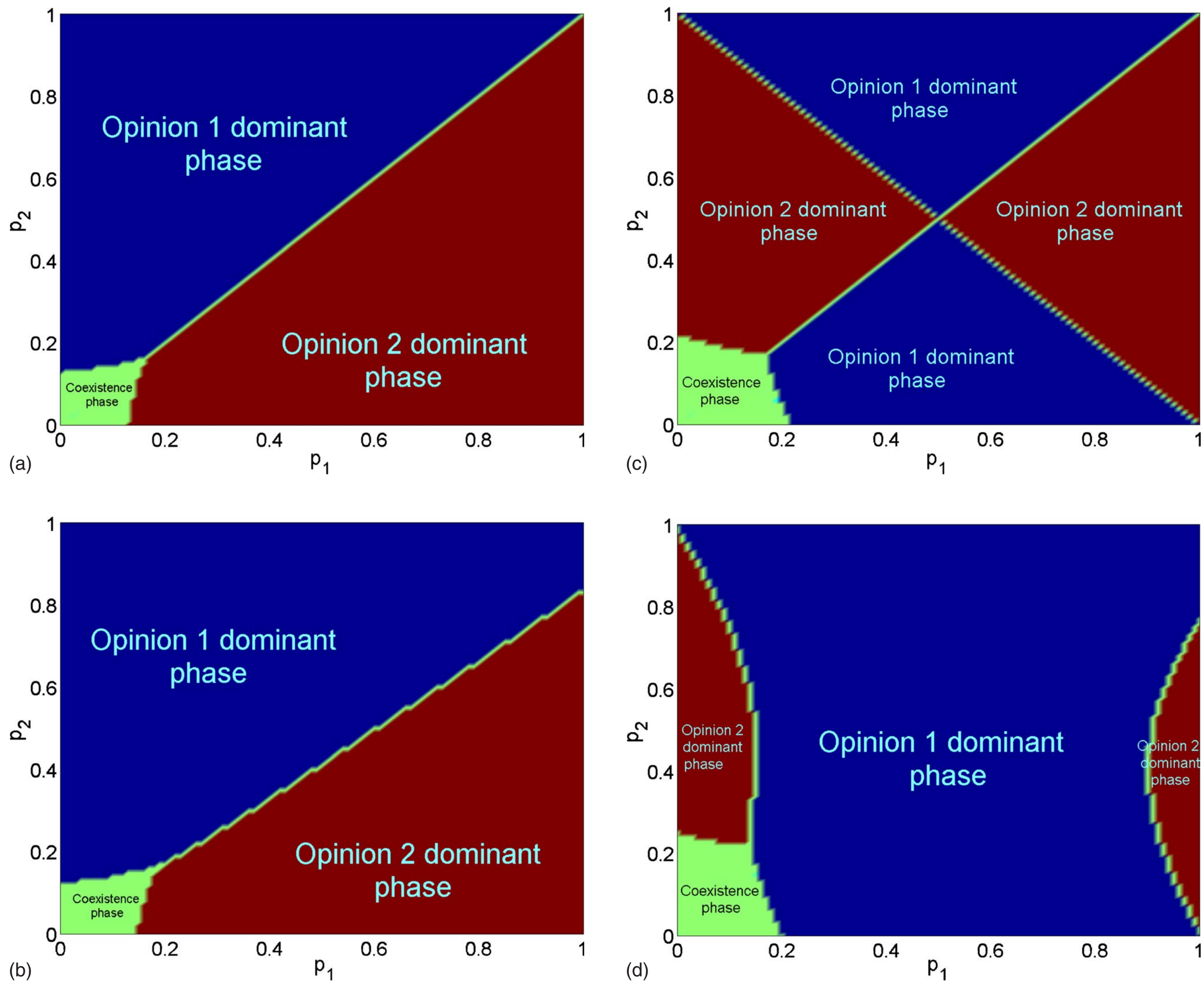

FIG. 3. (Color online) The phase diagrams for opinion formation on the plane of parameters p_1 and p_2 in class-adjacency-strength matrices. The class number $k=2$, and the population size $n_1=n_2=100$. The preference degrees for each class in the BSG payoff matrices: (a) and (c) $b_1=b_2=0.75$; (b) and (d) $b_1=0.80$, $b_2=0.75$. For (a) and (b), the interacting and updating graphs are identical, and for (c) and (d), the updating graph is the reverse of the interacting graph, where all the links' directions are reversed.

coexistence phase and dominance phase, respectively. As expected, the relative lowly clustering connections between congeneric individuals (whose strength can be noted by p_1 and p_2, respectively) are easily convinced by others during the process of opinion formation.

To observe the phase transition of opinion consensus and coexistence, we introduce an order parameter ρ denoting the percentage of individuals who factually hold the opinions consistent with their preferences when the (quasi)stationary opinion distribution is reached:

$$\rho = \frac{1}{n}\sum_{i=1}^{n} \delta_{f_i,p_i}, \tag{8}$$

where f_i is the factual opinion of the ith individual and p_i is his preference. If $\rho=1$, then all the individuals hold their respective preferences satisfactorily.

We first consider the opposite-structured case, where two classes of individuals have the equal-densely-clustering community structures with connection strengths $p_1=p_2$. Denote the class-adjacency-strength matrix of the interacting and updating graphs by

$$W = \begin{pmatrix} 1-p & p \\ p & 1-p \end{pmatrix},$$

and thus the modularity $\mathcal{M}=\frac{1}{2}-p$. Varying the tunable parameter p from 0 to 1 to observe the order parameter ρ, we find there exists a critical point $\mathcal{M}_c \approx 0.28$ that separates the opinion coexistence and dominance phases: As shown in Fig. 2, when $\mathcal{M}<\mathcal{M}_c$, ρ remains constant around 0.5 approximately, which shows that one opinion overwhelms the other, and the coexistence of different opinion clusters does not occur. When $\mathcal{M}>\mathcal{M}_c$ approaching the upper bound 0.5, the

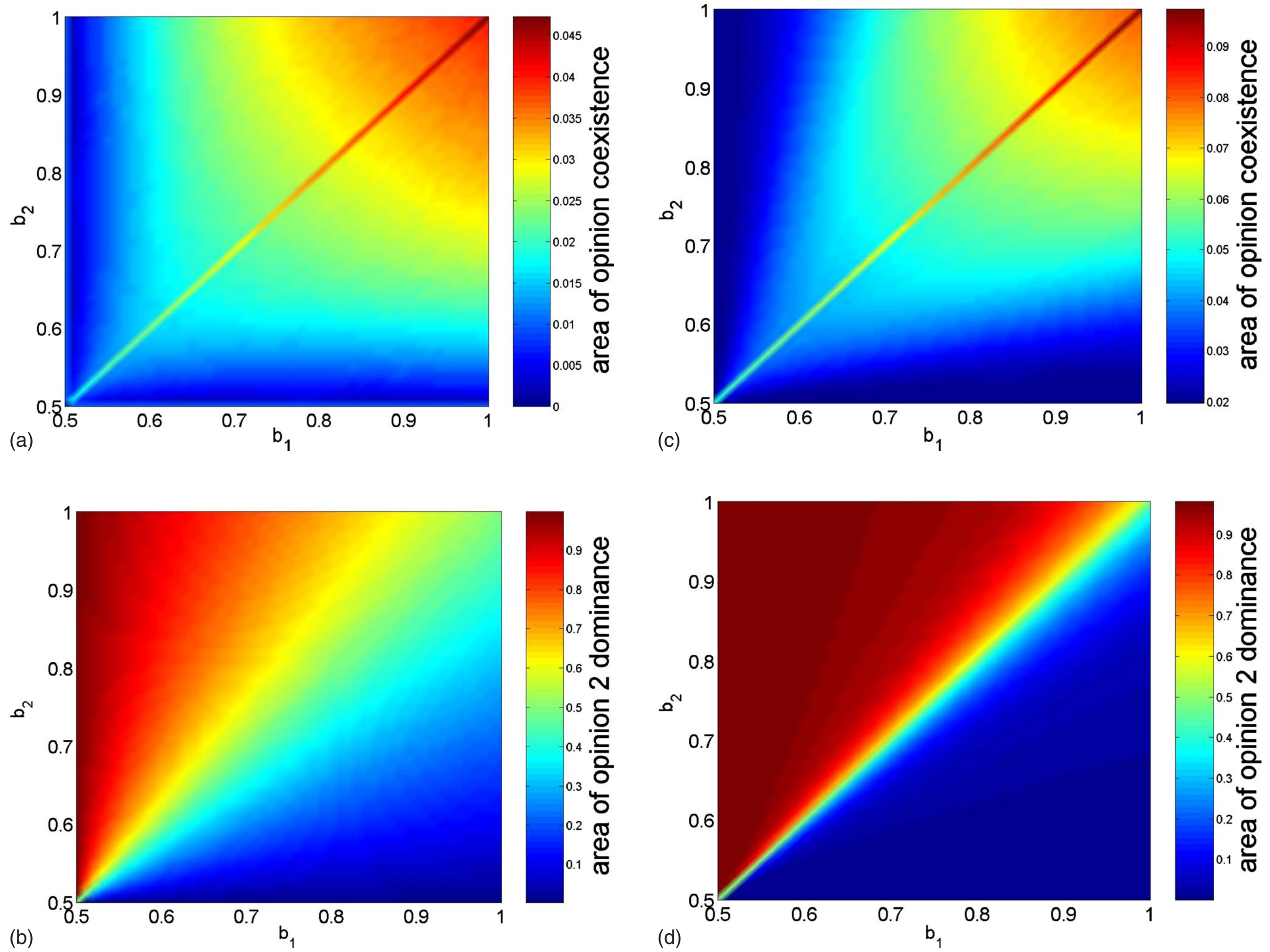

FIG. 4. (Color online) The correlation between the preference degree and the factual opinion distribution. The areas of opinion coexistence or dominance phases in the phase diagram are utilized to characterize the effects of preference degrees on the transition criticality of opinion formation. The areas of the coexistence phase region are color-coded in the right panel of (a) and (c); The areas of the opinion 2 dominant phase region are color coded in the right panel of (b) and (d). The class number $k=2$, and the population size $n_1=n_2=100$. For (a) and (b) the interacting and updating graphs are identical, and for (c) and (d) the updating graph is the reverse of the interacting graph.

modular feature of congeneric individuals becomes distinct gradually, the congeneric individuals constitute a community of the network, and their preferred opinions can be sustained and coexist with others. Particularly, $\mathcal{M}=0.5$ indicates that the network becomes two isolated communities corresponding to the two classes of individuals and $\rho=1$ shows a complete satisfaction that each individual can adopt the preference opinion, respectively.

We next consider the general case of a network with an arbitrary class-adjacency-strength matrix

$$W = \begin{pmatrix} 1-p_1 & p_1 \\ p_2 & 1-p_2 \end{pmatrix},$$

where p_1 and p_2 constitute the unit square on the parameter plane for the phase transition of opinion formation. As illustrated in Fig. 3, the final factual opinion distribution has three phases: all individuals reach a consensus in opinion 1 (and 2), and the mixed opinions coexist, which correspond to three phases: "opinion 1 (and 2) dominant phase" and "coexistence phase." Notice that the coexistence phase lies commonly in the area near the origin of the parameter plane, where both p_1 and p_2 are small and the modularity is relatively large, implying that the highly modular network structure of congeneric individuals is prone to form coexisting opinions. With the identical interacting and updating graphs, we denote the critical values of p_1 and p_2 separating the coexistence phase and the opinion 1 (or 2) dominant phase by p_1^{c1} and p_2^{c1} (or p_1^{c2} and p_2^{c2}). As shown in Figs. 3(a) and 3(b), p_1^{c1} (or p_2^{c2}) is an increasing function of p_2^{c1} (or p_1^{c2}), which indicates an oppositional relationship between the coexisting noncongeneric individuals: if one class of individuals increase the strength of their inner connections, the other class of individuals also need to enhance their community strength to maintain the coexistence and avoid being convinced. Similar is the relationship between the critical values p_1^{12} and p_2^{12} to separate the opinion 1 and 2 dominant phases. But when individuals follow different interacting and updating connections, as a demonstration given in Figs. 3(c) and 3(d), for example, the above-mentioned relationship could be broken and p_1^{12} has an extremely complicated dependence of p_2^{12}, which shows a complexity of the effects of topological structures on the dynamical phenomena on networks.

As shown in Figs. 3(a) and 3(b), the individuals' preference degrees lead to different phase diagrams. In the case of two classes with an identical preference degree, the phase diagram is symmetrical with respect to the line $p_1 = p_2$. Enhancing the preference degree of one opinion, we observe that the region of the corresponding opinion's dominant phase enlarges. To visualize and describe the effect of opinion preference degree on the opinion formation, with different preference degrees have we compared the area of phase regions in the above-mentioned phase diagrams on the p_1-p_2 parameter plane. We observe two classes of individuals with the preference degrees b_1 and b_2 varying from 0.5 to 1, respectively, and plot the areas of the coexistence phase region [in Figs. 4(a) and 4(c)] and opinion-2-dominant phase region [in Figs. 4(b) and 4(d)] with color-coded indication. The larger the difference between the preference degrees is, the more apt the multiclass individuals are to reach a consensus in the dominant opinion. We find that the strictly equal preference degrees obviously avail to the coexistence of mixed opinions, as shown in Figs. 4(a) and 4(c). Figures 4(b) and 4(d) reveal an approximately bilinear correlation between the preference degrees and the dominant phase area, and the area of the dominant phase is positively linearly correlated with the preference degree of the individuals who prefer it and negatively linearly correlated with the individuals who prefer the other.

V. CONCLUSION

To summarize, in this paper we have explored the opinion formation on networks, where individuals are classified according to their preferences and their interactions are characterized by two-person games with different payoff matrices. We have discussed the opinion formation among a population of individuals with the birth-death or death-birth updating rules in the evolutionary game theory literature and focused our main attention on the phenomenon of coexisting strategies and opinions. We have analytically shown that the general coexistence stability of strategy distributions for multiclass individual game systems concerns the network structures. We have also numerically studied the network's modular structural effects on the opinion dynamics, and the transition criticality between the consensus-opinion phase and mixed coexisting-opinions phase. The simulation results reveal a close positive correlation between the coexistence stability and the modularity of networks.

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[7] Consider a simple directed network model based on the ER random graph [8] as follows: Partition n vertices of the network into k classes (with the population size n_i, respectively), and introduce two wiring probability p_{in}^c and p_{out}^c for each class c. Connect a pair of vertices (v_i, v_j) with probability p_{in}^c if v_i and v_j belong to class c; otherwise, with probability p_{out}^c if the vertex v_j is not congeneric with v_i which belongs to other class than c. Thus the class-adjacency-strength matrix can be easily determined by the above parameters. If the network has two equal-sized classes explicitly,

$$W = \begin{pmatrix} p_{in}^1/(p_{in}^1 + p_{out}^1) & p_{out}^1/(p_{in}^1 + p_{out}^1) \\ p_{out}^2/(p_{in}^2 + p_{out}^2) & p_{in}^2/(p_{in}^2 + p_{out}^2) \end{pmatrix}.$$

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[9] Reference [10] introduced the modularity $\mathcal{M} = \Sigma_{c=1}^k \left[\frac{l_c}{m} - \left(\frac{d_c}{2m}\right)^2\right]$ for a partition of undirected networks. Note that only the out-degrees of vertices are calculated in our directed network model; thus, the modularity makes a difference in the

second term (because in undirected networks each link is just counted twice when summing the degrees of the vertices, while links are counted exact one time in directed networks). For the networks constructed with the tunable community-strength directed network model [7], we have the modularity $\mathcal{M}=\Sigma_{c=1}^{2}\left[\frac{n_c^2 p_{in}^c}{m}-\left(\frac{n_c^2 p_{in}^c+n_c n_{3-c} p_{out}^c}{m}\right)^2\right]$, where $m=\Sigma_{c=1}^{2}[n_c^2 p_{in}^c + n_c n_{3-c} p_{out}^c]$. Here, self-loops are allowed. In our simulations, we consider the two classes have the equal size of population/vertices ($n_1=n_2$) and the equal number of links ($p_{in}^1+p_{out}^1=p_{in}^2+p_{out}^2$); thus, the modularity is reduced to $\mathcal{M}=\frac{(p_{in}^1+p_{in}^2)}{2p}-\frac{1}{2}$ or $\mathcal{M}=\frac{1}{2}-\frac{p_1+p_2}{2}$, where p_1 and p_2 are parameters of the class-adjacency-strength matrices

$$W=\begin{pmatrix} 1-p_1 & p_1 \\ p_2 & 1-p_2 \end{pmatrix}.$$

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